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Quantum Mechanics on Noncommutative Riemann Surfaces

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Abstract

We study the quantum mechanics of a charged particle on a constant curvature noncommutative Riemann surface in the presence of a constant magnetic field. We formulate the problem by considering quantum mechanics on the noncommutative AdS_2 covering space and gauging a discrete symmetry group which defines a genus-g surface. Although there is no magnetic field quantization on the covering space, a quantization condition is required in order to have single-valued states on the Riemann surface. For noncommutative AdS_2 and subcritical values of the magnetic field the spectrum has a discrete Landau level part as well as a continuum, while for overcritical values we obtain a purely noncommutative phase consisting entirely of Landau levels.

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1 Introduction

Noncommutative quantum field theories have been studied very intensely over the last few years especially because of their relation to M-theory compactifications [1] and string theory in nontrivial backgrounds [2, 3, 4]. They are interesting because they preserve some of the nonlocal properties inherent in string theory. For example, T-duality is a manifest symmetry [5, 6]. (For recent reviews of noncommutative gauge theory see [7].) Recently, noncommutative Chern-Simons was shown to give an alternative description of the fractional quantum Hall effect [8, 9, 10, 11, 12].

At low enough energies the single-particle sector becomes relevant and thus it is enough to consider noncommutative quantum mechanics. (For early studies of noncommutativity in quantum mechanics see [13, 14, 15, 16, 17, 18].) In particular, one can consider the quantum mechanics of a charged particle moving on a two dimensional noncommutative surface in the presence of a constant magnetic field. The problem on the plane and the sphere has been considered in [19, 20, 21], on the noncommutative torus in [22] and on noncommutative AdS_2 in [23].

In this paper we generalize this to higher genus noncommutative Riemann surfaces. In Section 2 we review noncommutative $U(1)$ gauge theory on AdS_2. We study the quantum mechanics of a charged particle on noncommutative AdS_2 in a constant magnetic field in Section 3. his problem has also been considered in [23]; however, because only representations of the Lie algebra $sl(2,\mathbb{R})$ which integrate to representations of the group $SL(2,\mathbb{R})$ were used, a quantization of the magnetic field resulted. Such a quantization is certainly not observed in the commutative limit, since the topology of AdS_2 is trivial. We show that more general representations are allowed such that the magnetic field is not quantized.

We also discuss the energy spectrum. Unlike the usual Landau levels on the plane, for commutative AdS_2 the Hamiltonian has both a discrete spectrum and a continuum. Semi-classically this can be understood as follows: on a plane, for any finite energy the classical orbits are closed and single valuedness of the wave function phase around the orbit leads to a quantization of the energy. On AdS_2, if the energy is above a threshold, we have open trajectories and no quantization of the energy. The spectrum for noncommutative AdS_2 is similar, except that or magnetic field abobe a critical value

$B_{crit} = 1/\theta$ all motion is bounded and there is only a discrete spectrum.

In Section 4 we construct quantum mechanics on a noncommutative Riemann surface by modding AdS_2 by a discrete subgroup of $SO(2,1)$ which defines the cycles of a genus-g surface. Gauging of this discrete subgroup is just the requirement that the Hilbert space is projected to states that transform trivially under the action of the subgroup, which corresponds to invariance of (scalar) wavefunctions around the cycles, up to gauge transformations and vacuum angles. We show that this gauging requires a certain quantization condition for the magnetic field and demonstrate that in the commutative limit this condition reduces to the standard Dirac quantization of the flux. The Landau level spectrum for a noncommutative Riemann surface is the same as that of AdS_2 but with finite degeneracy. One also expects a discrete spectrum above the threshold, but little is known about this even in the commutative case. A partial list of studies of the same problem on a commutative Riemann surface is [24, 25]. The concept of a noncommutative Riemann surface was also discussed in [26].

Finally, in the last section we briefly discuss some open issues for future investigation.

2 Gauge theory on the noncommutative AdS_2

In this section we discuss $U(1)$ gauge theory on the noncommutative AdS_2. We follow closely the treatment of the noncommutative sphere in [21]. Field theory on the noncommutative sphere was introduced in [27] and studied rather extensively in [28]. First consider the Lie algebra

$$[x_i, x_j] = i\,\frac{\theta}{r}\,\epsilon_{ij}{}^{k} x_k \ , \tag{1}$$

where θ and r are real parameters which we take to be positive, $\epsilon_{123} = 1$ and indexes are raised and lowered with the metric $\eta = \mathrm{diag}(1, 1, -1)$. The rescaled generators $R_i = \frac{r}{\theta} x_i$ satisfy the $sl(2,\mathbb{R})$ relations

$$[R_i, R_j] = i\,\epsilon_{ij}{}^{k} R_k \ , \tag{2}$$

with the quadratic Casimir

$$R^2 = R_1^2 + R_2^2 - R_3^2 \ . \tag{3}$$

Let us briefly describe the unitary representations of $sl(2,\mathbb{R})$. These representations are infinite dimensional since the metric is of indefinite signature. Usually, in the mathematical literature [29] one finds the description of the representations of the Lie algebra which can be integrated to true representations of the groups $SL(2,\mathbb{R})$ or $SO(2,1)$. While somewhat less familiar than the unitary representations of $su(2)$, they can nevertheless be obtained exactly in the same way. One starts with an arbitrary R_3 eigenstate $|m\rangle$ of unit norm and obtain other states in the representation by applying $R_\pm = R_1 \pm iR_2$. Using the fact that R_i are hermitian, one can calculate the norm of these states and require it to be positive. After this analysis [30, 31], one obtains representations which are of the following types:

- *Principal discrete series:* These representations act on the Hilbert space

  $$\mathcal{D}_j^\pm = \{|j;m\rangle;\ m = \pm j, \pm j \pm 1, \pm j \pm 2, \dots\}\ .$$

  The state $|j;m\rangle$ has $R_3 = m$, and the state $|j;-j\rangle$ has the highest weight in $\mathcal{D}_j^-$ while $|j;-j\rangle$ has the lowest weight in $\mathcal{D}_j^+$. The Casimir equals $R^2 = j(1-j)$ where j is an arbitrary positive *real* number.

- *Principal continuous series:* These representations act on the Hilbert space

  $$\mathcal{C}_j^\alpha = \{|j,\alpha;m\rangle;\ m = \alpha, \alpha \pm 1, \alpha \pm 2, \dots\}\ .$$

  labeled by two continuous parameters j and α. The Casimir is given by $R^2 = j(1-j)$ for $j = 1/2 + is$ where s is real and positive. The parameter α is real and can be chosen to satisfy $\alpha \in [0,1)$. The states have $R_3 = m$.

- *Complementary continuous series:* These representations act on the Hilbert space

  $$\mathcal{E}_j^\alpha = \{|j,\alpha;m\rangle;\ m = \alpha, \alpha \pm 1, \alpha \pm 2, \dots\}\ .$$

  The parameter α is real and can be chosen to satisfy $\alpha \in [0,1)$ while j is real in the interval $j \in (1/2,1)$ and must satisfy $j(1-j) > \alpha(1-\alpha)$.

- *Identity representation:* This is the trivial one dimensional representation.

The representations in the discrete series form a discrete set only if we require them to integrate to representations of either the group $SL(2,\mathbb{R})$ or $SO(2,1)$. Then, j must be an integer or half integer for $SL(2,\mathbb{R})$, while for $SO(2,1)$ it must be an integer. In general, a unitary representation of a semi-simple Lie algebra is also a unitary representation of the universal covering group $\tilde{G}$ of all the groups G with the given algebra. Since such a group G has the form $G = \tilde{G}/\Gamma$ where Γ is a discrete subgroup of $\tilde{G}$, to obtain representations of G we must restrict to Γ-invariant representations of $\tilde{G}$. Equivalently, a necessary and sufficient condition for a representation of a semi-simple Lie algebra to integrate to a representation of the Lie group G, is to be a good representation of a maximal compact subgroup of G. Regarded as a Riemannian manifold (with the metric given by the Killing metric), the universal covering group of $SL(2,\mathbb{R})$ or $SO(2,1)$ is in fact the familiar AdS_3 of unit radius and nonperiodic time[a]. It has the topology $D \times \mathbb{R}$, where D denotes a disk. We can obtain $SL(2,\mathbb{R})$ by identifying time with period 4π and $SO(2,1)$ by identifying time with period 2π. Both groups have the topology $D \times S^1$. This leads to the quantization of j described above.

Noncommutative AdS_2 of radius r is defined as the matrix algebra generated by x_i in the $\mathcal{D}_j^+$ irreducible unitary representation where the Casimir satisfies

$$x^2 = x_1^2 + x_2^2 - x_3^2 = -r^2 \ ,$$

and x_3 is positive definite. We must take $j > 1$ so that x^2 be negative. Then the parameter θ is given by

$$\theta = \frac{r^2}{\sqrt{j(j-1)}} \ . \tag{4}$$

For states with $x_1, x_2 \sim 0$, $x_3 \simeq r$, (1) reduces to the planar noncommutativity relation $[x_1, x_2] = -i\theta$ and thus θ is identified as the noncommutativity parameter. Note that for fixed r, since j can vary continuously, there is no quantization of θ.

In the operator approach, scalar fields on noncommutative AdS_2 space are defined as arbitrary operators on the Hilbert space and thus can be identified with arbitrary elements of the algebra ψ. We can implement the infinitesimal action of $sl(2,\mathbb{R})$ on the generators of the noncommutative AdS_2 as $[R_i, x_j] = i\ \epsilon_{ij}{}^k x_k$. Since this action is a

[a]For an illuminating discussion of $SL(2,\mathbb{R})$ and its covering group see [32].

derivation, we can define it also on an arbitrary element ψ of the algebra as

$$L_i(\psi) = [R_i, \psi] \ .$$

We can then define the derivative operators $\nabla_i = -\frac{i}{r} R_i$ on ψ, which satisfy

$$[\nabla_i, \nabla_j] - \frac{1}{r}\epsilon_{ij}{}^k \nabla_k = 0 \ .$$

We now formulate gauge theory on the noncommutative AdS_2. The covariant derivative operators can be defined as a perturbation of the derivative operators

$$D_i = \nabla_i + iA_i \ .$$

Under gauge transformations, which are just time-dependent infinite dimensional unitary matrices U, the covariant derivative operators transform as

$$D'_i = UD_iU^{-1} \ . \tag{5}$$

It is convenient to also introduce covariant coordinates

$$X_i = i\theta D_i = x_i - \theta A_i \ ,$$

parametrizing a noncommutative two-dimensional membrane. The requirement that there be only two independent components of the gauge field on AdS_2 is equivalent to the requirement that there be no transversal excitations of the membrane. So the X_i satisfy the hyperboloid condition $X^2 = -r^2$, or, equivalently,

$$D^2 = \left(\frac{r}{\theta}\right)^2 = \frac{j(j-1)}{r^2} \ . \tag{6}$$

This can be rewritten as

$$x^i A_i + A_i x^i - \theta A^2 = 0 \ . \tag{7}$$

In the commutative limit $\theta \to 0$, (7) is just the condition that A_i is tangent to the hyperboloid.

We can define a gauge covariant field strength as

$$iF_{ij} = [D_i, D_j] - \frac{1}{r}\ \epsilon_{ij}{}^k D_k \ .$$

Notice that $F_{ij} = 0$ for vanishing A_i or any other gauge equivalent configuration. For a commutative time we also introduce $D_0 = \partial_0 + iA_0$ and define

$$iF_{0i} = [D_0, D_i] \ .$$

Since the integral on AdS_2 is just $\int \psi = 2\pi\theta \, \mathrm{Tr}(\psi)$ the Maxwell action takes the form

$$\mathcal{S} = -\frac{1}{4g^2} \int dt \, 2\pi\theta \, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \ .$$

3 Quantum mechanics and spectrum on noncommutative AdS_2

In this section we discuss the quantum mechanics of a charged particle in a constant magnetic field on a noncommutative AdS_2.

The magnetic field, defined as $B_i = \frac{1}{2}\,\epsilon_i{}^{jk}F_{jk}$, takes the form

$$iB_i = \epsilon_i{}^{jk} D_j D_k + \frac{1}{r} D_i \ . \tag{8}$$

To have a uniform magnetic field we will take B_i proportional to the gauge-covariant coordinate X_i

$$B_i = -\frac{B}{r} X_i = -\frac{i\theta B}{r} D_i \ ,$$

and this together with equation (8) implies

$$[D_i, D_j] = \frac{1-\theta B}{r} \, \epsilon_{ij}{}^k D_k \ ,$$

which, up to a rescaling of D_i, are just the $sl(2, \mathbb{R})$ relations. Thus we have

$$D_i = -i \, \frac{1-\theta B}{r} K_i \ , \tag{9}$$

where K_i satisfy the algebra (2). Since D_i still have to satisfy (6), we take the representation of K_i to be irreducible and of the form $\mathcal{D}_s^{\pm}$ with $s > 1$. We will show shortly that the choice of $\mathcal{D}_s^{+}$ or $\mathcal{D}_s^{-}$ depends on the value of B. By a gauge transformation we can bring the K_i in the standard form where K_3 is diagonal. The relation (6) implies that s must satisfy

$$(1-\theta B)^2 = \frac{j(j-1)}{s(s-1)} \ . \tag{10}$$

Since neither j nor s are quantized when considering unitary representations of the Lie algebra, the relation (10) does not imply any quantization of B as was assumed in [23]. This result is compatible with the commutative limit where B is not quantized, since AdS_2 has a trivial topology.

For a charged field ψ, with the gauge transformation $\psi' = U\psi$, we define the covariant derivative action as

$$D_i(\psi) = D_i\psi - \psi\nabla_i \ .$$

On the right hand side, D_i represents an element of the algebra while on the left hand side it denotes an action on ψ. We can also write this as

$$iD_i(\psi) = \frac{1}{r}(\gamma K_i\psi - \psi R_i) \ , \tag{11}$$

where $\gamma = 1-\theta B$.

Note that ψ is a matrix multiplied on the left by $\mathcal{D}_s^\pm$ representation matrices and on the right by $\mathcal{D}_j^+$ representation matrices. It is more convenient to have both of these multiplications described as actions on the left. Since the generators are hermitian, transposition is equivalent to complex conjugation and this takes $\mathcal{D}_j^+$ into $\mathcal{D}_j^-$. Concretely, to the matrix ψ_{nm} we associate the state

$$|\psi\rangle = \sum_{n,m=0}^{\infty} \psi_{mn}|s+m\rangle_s^\pm|-j-n\rangle_j^- \ ,$$

and then the relation (11) can be written as

$$iD_i|\psi\rangle = \frac{1}{r}(\gamma\mathcal{R}_i^{(s)\pm} + \mathcal{R}_i^{(j)-})|\psi\rangle \ ,$$

where $\mathcal{R}_i^{(s)\pm}$ ($\mathcal{R}_i^{(j)-}$) denote operators acting on states of the $\mathcal{D}_s^\pm$ ($\mathcal{D}_j^-$) representations. In this notation, the action of the generators J_i of the $sl(2,\mathbb{R})$, representing the infinitesimal symmetry of AdS_2 , takes the form

$$J_i|\psi\rangle = (\mathcal{R}_i^{(s)\pm} + \mathcal{R}_i^{(j)-})|\psi\rangle \ . \tag{12}$$

In particular, J_3 can be identified with angular momentum around the origin.

The equation of motion for ψ can be obtained from an action of the Schröedinger type

$$\mathcal{S} = \int dt\, 2\pi\theta\, \mathrm{Tr}\left(i\psi^\dagger\dot{\psi} + \frac{1}{2}D_i(\psi)^\dagger D^i(\psi)\right) \ . \tag{13}$$

Then the Hamiltonian is given by $H = -\frac{1}{2}D^2$, and with a little bit of algebra it can be rewritten as

$$H = \frac{\gamma}{2r^2}\left(J^2 + \left(\frac{Br^2}{\gamma}\right)^2\right) \ . \tag{14}$$

The spectrum and eigenstates of the Hamiltonian are trivially related to those of J^2, and thus they are given by pure representation theory. They can be obtained from the following tensor product decompositions

$$\mathcal{D}_s^- \otimes \mathcal{D}_j^- = \sum_{m=0}^{\infty} \mathcal{D}_{s+j+m}^- \ , \tag{15}$$

$$\mathcal{D}_s^+ \otimes \mathcal{D}_j^- = \sum_{n\in I} \mathcal{D}_{\alpha+n}^{\pm} \oplus \int_{s=0}^{\infty} \mathcal{C}_{1/2+is}^{\alpha} \ , \tag{16}$$

where $I = \{n \in \mathbb{Z}; 1/2 < \alpha + n \leq |s-j|\}$. In (16) the + sign is taken for $s > j$ and $\alpha = |s-j|\mathrm{mod}(1) \in [0,1)$. Note that in (16), the rhs. contains representations from both the discrete and the continuous series, and that the discrete series start at $k \geq \frac{1}{2}$.

To choose between $\mathcal{D}_s^+$ and $\mathcal{D}_s^-$ we require that the Hamiltonian (14) be bounded from below. For $B < 1/\theta$, since γ is positive we choose $\mathcal{D}_s^+$. By (16) there is only a finite number of discrete series representations and because of the second term in (14) the Hamiltonian is positive definite. The spectrum consists of a finite set of discrete Landau level energies

$$E_n = \frac{\gamma}{2r^2}\left((\alpha+n)(1-\alpha-n) + \left(\frac{Br^2}{\gamma}\right)^2\right) \ , \ n \in I \ ,$$

above which there is a continuous spectrum, starting at the threshold energy

$$E_{thres} = \frac{\gamma}{8r^2} + \frac{B^2r^2}{2\gamma} \tag{17}$$

For $B > 1/\theta$, since γ is negative, we have a Hamiltonian bounded from below if we choose $\mathcal{D}_s^-$. In this case there is only a discrete energy spectrum given by

$$E_n = \frac{\gamma}{2r^2}\left((j+s+n)(1-j-s-n) + \left(\frac{Br^2}{\gamma}\right)^2\right) \ , \ n = 0, \ldots, \infty \ .$$

This phase is a purely noncommutative one.

We can check that, in the limit $r^2 \to \infty$ with constant θ, the above spectrum reproduces the Landau levels on the noncommutative plane found in [20]. In that limit the continuous spectrum is pushed to infinity. For the discrete levels we have, up to $O(r^{-2})$ corrections,

$$j = \frac{r^2}{\theta} + \frac{1}{2} \ , \quad s = \frac{r^2}{|\gamma|\theta} + \frac{1}{2} \ , \quad E_n = (n + \frac{1}{2})|B|$$

in agreement with the planar result. The density of states agrees as well. This gives an independent justification for the choice of $\mathcal{D}_s^-$ for the representation of the covariant derivatives in the case $B > 1/\theta$, since the system maps to the correct overcritical planar phase. At the commutative AdS_2 limit, $\theta \to 0$ for constant r, we recover the standard results [24].

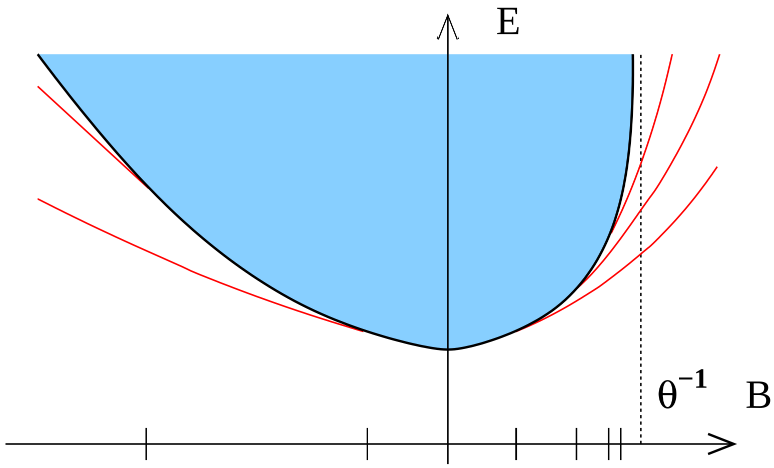

Figure 1: Continuous spectrum and Landau levels.

The form of the spectrum we obtained is depicted in Figure 1: for small positive values of the magnetic field ($|s - j| < \frac{1}{2}$) the spectrum is entirely continuous, with a threshold as in (17). For positive B corresponding to $s-j=\frac{1}{2}$, a single Landau level 'peels' from the bottom of the continuum. For $s-j=\frac{3}{2}$ a second level peels, and so on. As $B \to B_{crit}=1/\theta$, an infinity of Landau levels has formed, while the continuum is pushed to infinity. Above B_{crit} no more Landau levels are formed and there is no continuum. Similarly, for negative values of B, Landau levels peel from the continuum at points at which $s-j$ equals negative half-integers. Since $s>1$, there is a lowest such point, for $s-j=-[j-\frac{3}{2}]-\frac{1}{2}$ corresponding to some B_ℓ, at which the last Landau level peels. For $B<B_\ell$ no more Landau levels form. We see that, for negative B, there is a maximum number of Landau levels $N_{max}=[j-\frac{3}{2}]$. For large r or small θ, $N_{max}\sim\frac{r^2}{\theta}$. The entirely discrete spectrum above $B_{crit}=1/\theta$ and the existence of N_{max} are purely noncommutative effects.

4 Noncommutative Riemann surfaces

In this section we will formulate quantum mechanics on a noncommutative Riemann surface by gauging a discrete symmetry group of the action (13). To set the stage, we first review how to obtain a commutative Riemann surface endowed with a constant curvature metric by modding out the upper half-plane (or the mass hyperboloid) by the action of a Fuchsian group [33].

Consider a Riemann surface Σ of genus g on which we have chosen a canonical homology basis with generators $a_i,\ b_i,\ i=1,\dots,g$, i.e. the intersection numbers are given by

$$a_i\wedge a_j=0\ ,\ a_i\wedge b_j=\delta_{ij}\ ,\ b_i\wedge b_j=0\ . \tag{18}$$

Let us pick a representative in the homology class of each generator which also goes through a fixed point P on Σ. Then, a_i and b_i can be interpreted as generators of the fundamental group $\pi_1(\Sigma)$ based at P of the surface Σ. As such, they satisfy

$$\prod_{i=1}^{g}\left(a_ib_ia_i^{-1}b_i^{-1}\right)=1\ . \tag{19}$$

To understand the equation (19), take the above homology generators passing through P to be geodesics and then cut Σ along them. The resulting surface, called the cut Riemann surface Σ_c, is a $4g$-gon and the product on the lhs. of (19) is just the boundary cycle. This is obviously contractable to a point.

The group of isometries of the mass hyperboloid $x^2 = -r^2$ is $SO(2,1)$. Group elements of $SO(2,1)$ acting without a fixed point are called hyperbolic (they are called elliptic if they have a finite fixed point and parabolic if the fixed point is at infinity). Consider a discrete subgroup Γ of $SO(2,1)$ isomorphic to the fundamental group $\pi_1(\Sigma)$ and containing only hyperbolic elements. Then Γ must be generated by g_{a_i} and g_{b_i} satisfying

$$\prod_{i=1}^{g} \left(g_{a_i} g_{b_i} g_{a_i}^{-1} g_{b_i}^{-1} \right) = 1 \ . \tag{20}$$

All the nondegenerate Riemann surfaces of genus g can be obtained by modding out the mass hyperboloid by the action of such a group Γ. One can chose a covering of the hyperboloid such that each fundamental region is isomorphic to the cut Riemann surface Σ_c.

The action (13) is invariant under the infinitesimal $sl(2,\mathbb{R})$ transformations (12). These transformations, actually, involve both space translations and gauge transformations. They commute with the Hamiltonian and correspond to the magnetic translations of the particle. They are, thus, the appropriate transformations to be used in order to reduce the Hilbert space to the one of the genus-g Riemann surface. As we will see, if j and s are chosen appropriately, one can integrate the infinitesimal action of these generators and represent the group Γ on the set of states.

We therefore define quantum mechanics on the noncommutative Riemann surface as the system obtained by gauging the group Γ, in analogy to the commutative case. Since this group is discrete this just means that we must project onto the subspace of gauge invariant states. More generally, we can require invariance up to a phase (vacuum angle)

$$U(g_\alpha)\, \psi\, V^{-1}(g_\alpha) = e^{i\xi_\alpha}\, \psi \ , \ \alpha = 1, \dots , 2g \ . \tag{21}$$

In the above, the index α runs over the a_i and b_i cycles, while U and V denote the $\mathcal{D}_s^\pm$ and $\mathcal{D}_j^+$ representations of g_α. For $s = j$ the set of ψ's satisfying (21) form the algebra of "functions" on the noncommutative Riemann surface. For $s \neq j$ the set of ψ's satisfying (21) define a projective module which is the noncommutative analogue of the set of sections of a vector bundle.

Using (21) repeatedly we obtain the consistency condition

$$U(\prod_{i=1}^{g}\left(g_{a_i}g_{b_i}g_{a_i}^{-1}g_{b_i}^{-1}\right))\ \psi\ V^{-1}(\prod_{i=1}^{g}\left(g_{a_i}g_{b_i}g_{a_i}^{-1}g_{b_i}^{-1}\right))=\psi\ . \tag{22}$$

As we will now show, equation (22) implies a quantization of $\pm s-j$.

For j and s integers, $\mathcal{D}_s^{\pm}$ and $\mathcal{D}_j^{+}$ are also representations of $SO(2,1)$ and the relation (20) implies

$$U(\prod_{i=1}^{g}\left(g_{a_i}g_{b_i}g_{a_i}^{-1}g_{b_i}^{-1}\right))=V(\prod_{i=1}^{g}\left(g_{a_i}g_{b_i}g_{a_i}^{-1}g_{b_i}^{-1}\right))=1\ , \tag{23}$$

thus the consistency condition (22) is satisfied trivially. However, for j and s *real* positive, since the representations U and V are only representations of the universal covering group $\widetilde{SO(2,1)}$ we only have

$$U(\prod_{i=1}^{g}\left(g_{a_i}g_{b_i}g_{a_i}^{-1}g_{b_i}^{-1}\right))=e^{i\Theta_s^{\pm}}\ ,\ V(\prod_{i=1}^{g}\left(g_{a_i}g_{b_i}g_{a_i}^{-1}g_{b_i}^{-1}\right))=e^{i\Theta_j}\ , \tag{24}$$

as we will explain shortly. In this case, the consistency condition (22) is satisfied if the two phases in (24) are equal. The origin of the above phases is as follows: Since all the g_α are hyperbolic, they can be written as exponentials of elements in the Lie algebra. Using the exponential map, g_α can also be understood as group elements in the universal covering group $\widetilde{SO(2,1)}$. The product on the lhs. of (20) with the multiplication performed in the universal covering group does not necessarily give the identity but some element of $\widetilde{SO(2,1)}$ which projects to the identity of $SO(2,1)$. By looking at the form of the R_3 (or K_3) generators one can see that such an element is represented by a phase.

Let us associate to each g_α a curve in $SO(2,1)$ denoted $g_\alpha(t)$ representing a portion of a one dimensional subgroup passing through g_α such that $g_\alpha(0)$ is the identity and $g_\alpha(1)=g_\alpha$. Then to the product $\prod_{i=1}^{g}\left(g_{a_i}g_{b_i}g_{a_i}^{-1}g_{b_i}^{-1}\right)$ we associate a curve of length $4g$ by translating and joining the curves $g_\alpha(t)$ in the obvious way: for $t\in[0,1)$ the curve is given by $g_{a_1}(t)$; for $t\in[1,2)$ the curve is given by $g_{a_1}(1)g_{b_1}(t-1)$; and so on. Due to the relation (20) this must be a closed curve in $SO(2,1)$. However, the curve winds $2(g-1)$ around the noncontractable S^1 cycle of $SO(2,1)$ and thus it is an open curve in $\widetilde{SO(2,1)}$.

Before we calculate the winding in our problem, let us describe one way of obtaining it for an arbitrary closed curve $g(t)$ in $SO(2,1)$. Fix a reference point P on the hyperboloid and a reference tangent vector at P. The curve $g(t)P$ is a closed curve on the hyperboloid. The action of $g(t)$ on the reference vector gives a periodic vector field around the curve $g(t)P$. The winding is just the number of times the vector spins around itself as it goes once around the curve and is a topological invariant.

In our problem the curve $g(t)P$ is just the boundary of Σ_c, and the reference vector is parallel transported around the boundary of Σ_c. Under parallel transport on a hyperboloid of radius r around a closed loop enclosing an area A, a vector is rotated by an angle $\phi = A/r^2$. Since the scalar curvature is given by $R = -2/r^2$, using the Gauss-Bonnet theorem one can find the area of the surface Σ to be $A = 4\pi(g-1)r^2$. Thus under parallel transport around Σ a vector rotates an angle $\phi = 4\pi(g-1)$. Since we have $e^{iR_3\phi} = e^{i\Theta_j}$, the phase is given by $e^{i\Theta_j} = e^{4\pi i(g-1)j}$. The group Γ defined by the relation (20) is only represented projectively

$$V(\prod_{i=1}^{g}\left(g_{a_i}g_{b_i}g_{a_i}^{-1}g_{b_i}^{-1}\right)) = e^{4\pi i(g-1)j} \ . \tag{25}$$

Projective representations of Γ were also considered in [26] where they were obtained with the help of a gauge field on the Poincare plane. Here we see that projective representations naturally occur if j is not an integer or half-integer. Finally, the consistency condition (22) implies the quantization

$$\pm s - j = \frac{n}{2(g-1)} \ , \tag{26}$$

where n is an arbitrary integer.

From experience with the noncommutative sphere and torus we know that a more relevant quantity is a rescaled magnetic field $\widetilde{B} \equiv B(1-\theta B)^{-1}$. This would be the strength of the Seiberg-Witten mapped commutative gauge field in the planar case. From equation (10) we obtain

$$\widetilde{B} = \frac{1}{r^2}\left(\pm\sqrt{s(s-1)} - \sqrt{j(j-1)}\right) \ . \tag{27}$$

Since j is fixed for a given r and θ by relation (4), and $\pm s - j$ is quantized as in (26) we see that $\widetilde{B}$ can only take discrete values. Note however that, unlike the commutative case, the values of $\widetilde{B}$ are not equally spaced.

As a check, consider the commutative limit, obtained by taking j and s to infinity while keeping r and B finite (we must choose $\mathcal{D}_s^+$). In this limit we have $B = \frac{1}{r^2}(s-j)$, thus we must keep $s-j$ finite. Using this, we obtain the following integral quantization for the flux

$$\Phi \equiv AB = 2\pi n \ . \tag{28}$$

This is the expected Dirac quantization (or integrality of the first Chern number).

5 Concluding remarks

We have formulated the problem of a charged particle on a noncommutative genus-g Riemann surface and found the condition required for the existence of scalar wavefunctions. The spectrum of the particle, on the other hand, has not been fully identified. To achieve this, we would need to identify the physical states which satisfy the genus-g condition (21). This is, in principle, a purely group-theoretic problem. We expect the degeneracy of each discrete Landau level to become finite, and also the continuous spectrum to be fragmented into discrete nondegenerate states. Carrying out this calculation and identifying the full spectrum and degeneracies is a very interesting open issue.

The Dirac-like quantization condition for the strength of the magnetic field was derived by demanding invariance of the wavefunction under magnetic translations around the cycles of the noncommutative Riemann surface. It should be stressed that, as in the noncommutative torus case and unlike the sphere, this is not a requirement for consistency of the problem. In fact, we could have promoted the wavefunction ψ into a multicomponent vector by tensoring it with an N-dimensional vector space V_N and demand invariance under combined magnetic translations and $U(N)$ transformations, which would have resulted in an N-fold decrease in the unit of quantization in (26). This corresponds to 'overlapping' N copies of the fundamental domain of the Riemann surface.

In the toroidal case [22], the problem can be analyzed entirely in the canonical framework by defining physical coordinate and momentum variables which are well-defined on the torus. The representation theory of the algebra of these observables reproduces the above extended wavefunctions. In the genus-g case there is no immediately obvious complete set of such observables. Formulating and analyzing the noncommutative Riemann problem in terms of such canonical observables is an interesting open problem.

Finally, we should remark that, although here we have only considered $AdS_2 = SL(2,\mathbb{R})/U(1)$, it is obvious that the construction can be generalized to G/H where G is a real semisimple Lie group and H is its maximal compact subgroup. Application of this technique to physically relevant situations, such as the noncommutative gravity setting of [34], would be an interesting possibility.

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